Cohomology of Projective Spaces

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Derived Functors

Knowing sheaf cohomology of things is very useful in solving problems in multi-variable complex analysis, classical and abstract algebraic geometry ......

We can define the sheaf cohomology in generality using the concept of derived functors.

Definition

Given an left exact functor $F : \mathfrak{A} \to \mathfrak{B}$ and some injective resolution $I^{\cdot}$ of X in $\mathfrak{A}$ with induced morphisms $d^i : F(I^i) \to F(I^{i+1})$, a right derived functor $R^i F$ is defined to be $\ker d^i / \operatorname{im} d^{i-1}$.

Of course this definition is also valid for left derived functors and projective resolutions etc.

Abelian Group Sheaf Cohomology and Mayer-Vietoris

Applying the above to the left exact global section functor $\Gamma : \mathfrak{Ab}(X) \to \mathfrak{Ab}$ of a topological space X, we can get our corresponding cohomology of sheaves of abelian groups $\mathrm{H}^i(X, \mathcal{F}) = R^i\Gamma$ from injective resolution $I^{\cdot}$ of sheaf $\mathcal{F}$ on X.

Computing cohomologies directly from an explicit injective resolution for every $(X, \mathcal{F})$ is usually difficult. But we can define the so-called Čech cohomology to help us. To motivate its definition, let's first consider the Mayer-Vietoris Sequences.

Theorem (Mayer-Vietoris Sequences)

If $X = U \cup V$, $\mathcal{F}$ a sheaf on X, and $i : U \to X$, $j : V \to X$, $k : U \cap V \to X$ are the inclusion maps, we get a long exact sequence of cohomologies $0 \to \mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(U, \mathcal{F}) \oplus \mathrm{H}^0(V, \mathcal{F}) \to \mathrm{H}^0(U \cap V, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{F}) \to \cdots$

Čech Cohomology

Likewise, we define the Čech cohomology of $(X, \mathcal{F})$ by using $\mathfrak{U} = \{U_i\}_{i \in I}$ an ordered open cover of X.

The cohomologies would be based on the cochain of complexes

$$C^p := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})$$

which forms a sequence $0 \to C^0 \to C^1 \to \cdots \to C^p \to \cdots$ with coboundary map $d^p : C^p \to C^{p+1}$ defined as

$$d^p(\ldots, \alpha_{i_0, \ldots, i_p}, \ldots)_{i_0, \ldots, i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, \hat{i}_k, \ldots, i_p}\Big|_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}}$$

Cohomology of Projective Spaces

For projective spaces, we have $X = \mathbb{P}^r_A = \operatorname{Proj} S$ where $S = A[x_0, \ldots, x_r]$ and a quasi-coherent sheaf $\mathcal{F} = \oplus_{n \in \mathbb{Z}} \mathcal{O}(n)$ where $\mathcal{O}(n) = \widetilde{S(n)}$.

This structure natually admits an (affine) open covering $\mathfrak{U} = \{\operatorname{Spec} S_{(x_i)}\}_{0 \le i \le r}$ where $S_{(x_i)} = A[\frac{x_0}{x_i}, \ldots, 1, \ldots, \frac{x_r}{x_i}]$ is the localized degree 0 coordinate ring. Now we have the Čech complexes

$$\begin{aligned} C^p &= \prod_{i_0 < \cdots < i_p} \mathcal{F}(\operatorname{Spec} S_{(x_{i_0} \cdots x_{i_p})}) \\ &= \prod_{i_0 < \cdots < i_p} \oplus_{n \in \mathbb{Z}} \mathcal{O}(n)(\operatorname{Spec} S_{(x_{i_0} \cdots x_{i_p})}) \\ &= \oplus_{n \in \mathbb{Z}} \prod_{i_0 < \cdots < i_p} \widetilde{S_{(x_{i_0} \cdots x_{i_p})}(n)} \\ &\cong \prod_{i_0 < \cdots < i_p} S_{x_{i_0} \cdots x_{i_p}} \end{aligned}$$

Computing the Čech cohomology

So we have the sequence

$$0 \to \prod_i S_{x_i} \xrightarrow{d^0} \prod_{i,j} S_{x_i,x_j} \xrightarrow{d^1} \cdots \xrightarrow{d^{r-2}} \prod_i S_{x_0,\ldots,\hat{x}_i,\ldots,x_r} \xrightarrow{d^{r-1}} S_{x_0,\ldots,x_r} \to 0$$

which induces the Čech cohomology sequence.

$$0 \to \check{\mathrm{H}}^0(X,\mathcal{F}) = \ker d^0 \to \check{\mathrm{H}}^1(X,\mathcal{F}) \to \cdots \to \check{\mathrm{H}}^i(X,\mathcal{F}) = \frac{\ker d^i}{\operatorname{im} d^{i-1}} \to$$

$$\cdots \to \check{\mathrm{H}}^{r-1}(X,\mathcal{F}) \to \check{\mathrm{H}}^r(X,\mathcal{F}) = \operatorname{coker} d^{r-1} \to 0$$

with which we can start our calculation...

Useful Facts

Theorem (1)

If X is affine and $\mathcal{F}$ a quasi-coherent sheaf, then $\mathrm{H}^p(X, \mathcal{F}) = 0 \ \forall p > 0$.

Theorem (2)

If X is noetherian separated scheme, $\mathfrak{U}$ an open affince cover and $\mathcal{F}$ a quasi-coherent sheaf, then $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) \to \mathrm{H}^p(X, \mathcal{F})$ *are isomorphisms for* $p \geq 0$.

Theorem (3)

For $Y \hookrightarrow X$ a closed subscheme with inclusion map i, we have the exact sequence $0 \to \mathfrak{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$

Theorem (4)

If a module M have an isomorphism to itself by multiplication of f, then $M_f = 0$ implies $M = 0$.

Table of $\mathrm{H}^p(\mathbb{P}^r_A, \mathcal{O}(n))$

n	$p=0$	$p=1$	$\dots$	$p=r-1$	$p=r$
m	$A(\dots, f_i, \dots)$	0	$\dots$	0	0
1	$A(x_0, \dots, x_r)$	0	$\dots$	0	0
0	A	0	$\dots$	0	0
-1	0	0	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-r$	0	0	$\dots$	0	0
$-r-1$	0	0	$\dots$	0	B
$-r-2$	0	0	$\dots$	0	$B(x_0^{-1}, \dots, x_r^{-1})$
$-r-1-m$	0	0	$\dots$	0	$B(\dots, f_i^{-1}, \dots)$

where $B = A(\frac{1}{(x_0 \cdots x_r)})$, $m > 1$, and f_i are the degree m monomials in $x_0, \dots, x_r$.

References

Robin Hartshorne (1977)
Algebraic Geometry

And most importantly my mentor: Zhu, Yuecheng!